

MEASUREMENT ISOMORPHISM OF GRAPHS

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ABSTRACT. The d -measurement set of a graph is its set of possible squared edge lengths over all d -dimensional embeddings. In this note, we define a new notion of graph isomorphism called *d -measurement isomorphism*. Two graphs are d -measurement isomorphic if there is agreement in their d -measurement sets. A natural question to ask is “what can be said about two graphs that are d -measurement isomorphic?” In this note, we show that this property coincides with the *2-isomorphism* property studied by Whitney.

1. INTRODUCTION

Given a graph Γ we can consider placing each vertex at some position in \mathbb{E}^d and then measuring the squared Euclidean length of each of the graph’s edges. This gives us the coordinates of a single “measurement point” in \mathbb{R}^e , where e is the number of edges in the graph. As we alter the vertex positions, the measurement point will typically change. The d -dimensional measurement set, $M_d(\Gamma)$, is the union of all achievable measurement points as we vary over all possible placements of the vertices in \mathbb{E}^d . Suppose that, after some permutation of the e coordinate axes, we have agreement in the d -dimensional measurement sets of two graphs, Γ and Δ . Then we say that the graphs Γ and Δ are d -measurement isomorphic.

Clearly, two isomorphic graphs must be d -measurement isomorphic. But the converse is not true. For example, the measurement set of *any* forest graph is the entire first octant of \mathbb{R}^e as there are no constraints on the achievable edge lengths. A natural question to ask is “what can be said about two graphs that are d -measurement isomorphic?” In this note, we relate this type of isomorphism to a graph property studied by Whitney [3] called *2-isomorphism*. Our main result is that for any d , two graphs are d -measurement isomorphic if and only if they are 2-isomorphic. In particular, for 3-connected graphs, this means that two graphs are d -measurement isomorphic if and only if they are isomorphic graphs.

Definition 1.1. A *graph* Γ is a set of v vertices $\mathcal{V}(\Gamma)$ and e edges $\mathcal{E}(\Gamma)$, where $\mathcal{E}(\Gamma)$ is a set of two-element subsets of $\mathcal{V}(\Gamma)$.

Definition 1.2. Two graphs Γ and Δ , are *isomorphic* if there is a bijection ϱ between $\mathcal{V}(\Gamma)$ and $\mathcal{V}(\Delta)$ such that $\{x, y\} \in \mathcal{E}(\Gamma)$ iff $\{\varrho(x), \varrho(y)\} \in \mathcal{E}(\Delta)$.

Next we define two weaker notions of graph isomorphism. These allow us to move around barely connected parts of the graph without changing the equivalence class.

Definition 1.3. A *cut vertex* of graph is a vertex whose removal disconnects the graph. A *split* operation breaks a cut vertex into two vertices to produce two disjoint subgraphs. Two graphs are *1-isomorphic* if they become isomorphic under a finite sequence of split operations. See Figure 1.

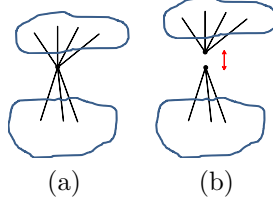


FIGURE 1. The split operation. The graphs (a) and (b) are 1-isomorphic.

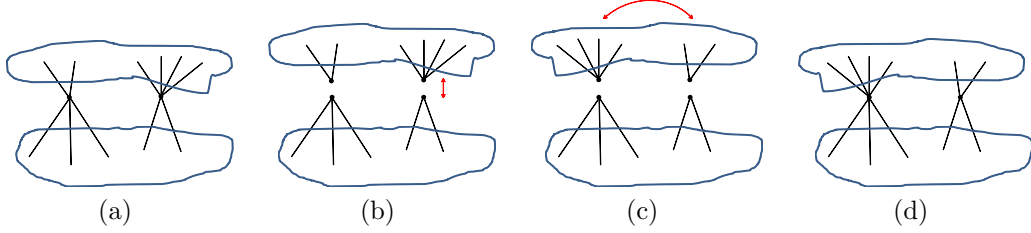


FIGURE 2. The reversal operation. The graphs, (a) and (d) are 2-isomorphic. Note that the edge lengths of the frameworks are unchanged.

Definition 1.4. (Following [2]) If $S \subset \mathcal{E}(\Gamma)$ then let $\Gamma[S]$ denotes the subgraph induced by S . A partition $\{S, T\}$ of $\mathcal{E}(\Gamma)$ is a *2-separation* of Γ if $|S| \geq 2 \leq |T|$ and $|\mathcal{V}(\Gamma[S]) \cap \mathcal{V}(\Gamma[T])| = 2$. Let $\{S, T\}$ be a 2-separation of Γ and let the *cut pair* $\mathcal{V}(\Gamma[S]) \cap \mathcal{V}(\Gamma[T])$ be $\{x, y\}$. Let Γ' be the graph obtained from Γ by interchanging in $\Gamma[S]$ the incidences of the edges at x and y . Then we say that Γ' is obtained from Γ by a *reversal* operation. Two graphs are *2-isomorphic* if they become 1-isomorphic after a finite sequence of reversals. See Figure 2.

Note that is not the same notion of 2-isomorphism studied in [4].

Remark 1.5. Since 3-connected graphs have no 2-separations, for such graphs 2-isomorphism coincides with graph isomorphism.

Next we define another notion of graph equivalence.

Definition 1.6. A *cycle* is a path of adjacent vertices that starts and ends at the same vertex, and with no vertex repeated in the path. Two graphs Γ and Δ , are *cycle isomorphic* if there is bijection σ between $\mathcal{E}(\Gamma)$ and $\mathcal{E}(\Delta)$ such that for any $S \subset \mathcal{E}(\Gamma)$, $\Gamma[S]$ is a cycle iff $\Delta[\sigma(S)]$ is a cycle.

In [3], Whitney proved the following theorem that will provide all of the heavy lifting that we will need in this note.

Theorem 1.7 (Whitney). *Two graphs are cycle-isomorphic iff they are 2-isomorphic.*

We now define some notions related to graph embeddings.

Definition 1.8. A *configuration* p of a vertex set \mathcal{V} is a mapping from \mathcal{V} to \mathbb{E}^d . Let $C^d(\mathcal{V})$ be the space of configurations of \mathcal{V} in \mathbb{E}^d . For $p \in C(\mathcal{V})$ and $u \in \mathcal{V}$, let $p(u)$ denote the image of u under p . A *framework* (p, Γ) is the pair of a graph and a configuration of its vertices. For a given graph Γ the *length-squared function* ℓ_Γ is the function assigning to each edge of Γ its squared edge length in the framework. That is, the component of $\ell_\Gamma(p)$ in the direction

of an edge $\{u, w\}$ is $|p(u) - p(w)|^2$. Once we fix an (arbitrarily) identification of each edge in $\mathcal{E}(\Gamma)$ with an associated coordinate axis in \mathbb{R}^e , we can interpret the length-squared function as being of the type: $\ell_\Gamma : C^d(\mathcal{V}) \rightarrow \mathbb{R}^e$.

Definition 1.9. The d -dimensional *measurement set* $M_d(\Gamma)$ of a graph Γ is defined to be the image in \mathbb{R}^e of $C^d(\mathcal{V})$ under the map ℓ_Γ . These are nested by $M_d(\Gamma) \subset M_{d+1}(\Gamma)$ and eventually stabilize by $M_{v-1}(\Gamma)$.

In our context of measurement sets of graphs, we define a new notion of isomorphism.

Definition 1.10. Two graphs, Γ and Δ , both with e edges, are d -*measurement isomorphic* if there is an identification of each edge in $\mathcal{E}(\Gamma)$ with an associated coordinate axis in \mathbb{R}^e , and an identification of each edge in $\mathcal{E}(\Delta)$ with an associated coordinate axis in \mathbb{R}^e , under which $M_d(\Gamma) = M_d(\Delta)$.

Our main result is the following

Theorem 1. *For any d , two graphs are d -measurement isomorphic iff they are 2-isomorphic.*

This also gives us the following:

Corollary 1.11. *For any two integers d_1 and d_2 , if a pair of graphs are d_1 -measurement isomorphic then they are d_2 -measurement isomorphic.*

Remark 1.12. Testing cycle isomorphism of graphs is as computationally difficult as testing graph isomorphism [1], and thus so is testing 2-isomorphism and d -measurement isomorphism.

2. PROOF

We will prove our theorem through a cycle of implications. For these arguments we first fix d as it turns out that our arguments do not depend on it.

2.1. 2-isomorphism \Rightarrow d -measurement isomorphism. The graphs Γ and Δ are 2-isomorphic if they become isomorphic after a finite number of splits and reversals. Clearly, a split operation does not change M_d .

Likewise, let $\{S, T\}$ be a 2-separation of Γ with cut pair $\{x, y\}$ and let Γ and Γ' be related by the reversal across this pair. Under reversal, there is a canonical bijection between $\mathcal{E}(\Gamma)$ and $\mathcal{E}(\Gamma')$. Let us fix our edge-axis identifications to be consistent with this bijection.

For any $p \in C^d(\mathcal{V}(\Gamma))$, we can reflect, in \mathbb{E}^d , the positions of the vertices of $\mathcal{V}(\Gamma[S]) \setminus \{x, y\}$ across the hyperplane bisecting the segment \overline{xy} to obtain a new configuration p' . Under this construction, we have $\ell_\Gamma(p) = \ell_{\Gamma'}(p')$. See Figure 2. Thus $M_d(\Gamma) = M_d(\Gamma')$ and they must be d -measurement isomorphic.

2.2. d -measurement isomorphism \Rightarrow cycle isomorphism. We start by showing that a cycle graph on k edges is not d -measurement isomorphic to any other graph.

Lemma 2.1. *Let c be a cycle of k edges, and b be any other graph with k edges. Then c and b are not d -measurement isomorphic.*

Proof. If a graph is a forest with k edges, there are no constraints on any of the achievable squared edge lengths, and thus its measurement set is the entire first octant of \mathbb{R}^k .

If a graph with k edges, is not a forest then it must have a cycle as a subgraph. In this case, its measurement set cannot be the entire first octant of \mathbb{R}^k since there is no framework (in any dimension) where all but one of the edges of the cycle has zero length.

Thus, if c is a cycle and b a forest, their measurement sets cannot agree under any edge-axis identifications.

If c is a cycle and b is neither a cycle or a forest, then b must have an edge whose removal does not turn b into a forest. Meanwhile the removal of any edge turns c into a forest. In terms of measurement sets, edge removal corresponds to projecting the measurement set onto coordinates associated with the appropriate $k - 1$ edges. These projections for b and c cannot not agree, as one produces the measurement set of a forest and one produces the measurement set of a non-forest. Since the projected measurement sets do not agree, the original measurement set cannot have agreed as well. \square

Suppose that Γ is not cycle isomorphic to Δ . Then for any edge bijection, σ , there must be an edge subset c of Γ such that $\Gamma[c]$ is a cycle, while $\Delta[\sigma(c)]$ is not a cycle.

Let us fix our edge-axis identifications to be consistent with σ . Next, let us project the measurement sets $M_d(\Gamma)$ and $M_d(\Delta)$ down to the subspace of \mathbb{R}^e corresponding to the edge set of c and $\sigma(c)$ respectively. In the case of Γ we will obtain the measurement set of a cycle, while for Δ we will obtain the measurement set of a non-cycle. These two measurement sets cannot be the same by Lemma 2.1. Thus $M_d(\Gamma)$ cannot be the same as $M_d(\Delta)$ under this edge-axis identification.

This is true for all edge-axis identifications consistent with the bijection σ . And, by assumption, this is true for all bijections. Thus it is true for all edge-axis identifications and Γ and Δ cannot be d-measurement isomorphic.

2.3. cycle isomorphism \Rightarrow 2-isomorphism. This is simply Theorem 1.7. And we are done.

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